Analysis of Slightly Anisotropic Shells

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A perturbation method of solution for the governing system of differential equations for laminated anisotropic shells is presented. These shells may be composed of an arbitrary number of bonded layers, each with a different thickness, different orientation, and different anisotropic elastic properties. Such a construction can appropriately describe filament-wound pressure vessels. By the perturbation scheme it is possible to reduce the system of anisotropic shell equations to successive systems of orthotropic shell equations. Thus, the complete solution consists of a series of solutions of equivalent orthotropic shells. The general perturbation system of equations is then specialized for cylindrical shells using the well-known Donnell approximations. The particular case of uniform pressurization and axial force is solved in detail.

Introduction

FILAMENT-WOUND pressure vessels are prominent in the aerospace industry because their improved strength-to-weight ratio permits the design of more efficient rocket motor cases. The construction and material properties of these pressure vessels are such that they can be appropriately characterized as laminated anisotropic shells. A theory governing the behavior of laminated anisotropic shells has recently been presented in Refs. 1 and 2. The mathematical model employed in the theory represents a shell structure composed of an arbitrary number of bonded layers with different thicknesses, orientations of elastic axes, and anisotropic elastic properties. A new feature arising from such a construction technique is the appearance of a coupled system of differential equations. This coupling reflects the simultaneous response of extensional and flexural deformations for a single load component (either an in-plane force or a bending moment).‡ As a consequence of this coupling and the general difficulty of solving anisotropic problems, very few solutions have appeared in the literature.

In Refs. 3 and 4 a perturbation scheme was employed to uncouple the governing system of equations. This method of analysis reduces the original system of equations to successive systems of homogeneous anisotropic shell equations. As a result, solutions from homogeneous shell theory can be used. In this paper the perturbation scheme is used to reduce general anisotropic shell equations to successive systems of orthotropic shell equations, provided that the degree of anisotropy is small.§ The final systems of equations are in a form used for analysis of nonhomogeneous, orthotropic shells with additional terms to account for anisotropy. The general system of equations is then specialized for circular cylindrical shells. The case of uniform pressurization of a semi-infinite cylindrical shell is presented as an illustration of practical application.

Recapitulation of Basic Equations

Let α, β, z be orthogonal space coordinates of the shell. The line element in this coordinate system is given by

$$ds^2 = A^2[1 + (z/R_1)]^2 d\alpha^2 + B^2[1 + (z/R_2)]^2 d\beta^2 + dz^2 \quad (1)$$

where A and B are surface metric coefficients, which, along with the principal radii of curvature R_1 and R_2, satisfy the Gauss-Codazzi relations.

The theory developed in Refs. 1 and 2 is predicated on the Kirchhoff-Love hypothesis on deformation. Application to thin shells requires that quantities of the form z/R_1 and z/R_2 be neglected in comparison with unity.

For slightly anisotropic shells, the stress resultant-displacement and the stress couple-displacement relations from Ref. 1 are re-examined.

$$\begin{bmatrix} N_\alpha \\ N_\beta \\ N_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ 2\chi_{12} \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} M_\alpha \\ M_\beta \\ M_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ 2\chi_{12} \end{bmatrix} \quad (3)$$

where $\epsilon_1, \epsilon_2, \gamma_{12}$ are the in-plane strains, χ_1, χ_2, χ_{12} are changes of curvature, and A_{ij}, F_{ij}, D_{ij} are elastic coefficients defined by

$$A_{ij} = \sum_{k=1}^{N} C_{ij}^{(k)}(h_k - h_{k-1})$$
$$F_{ij} = \frac{1}{2}\sum_{k=1}^{N} C_{ij}^{(k)}(h_k^2 - h_{k-1}^2) \quad (4)$$
$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} C_{ij}^{(k)}(h_k^3 - h_{k-1}^3)$$

The $C_{ij}^{(k)}$ coefficients represent the anisotropic elastic properties of the kth layer. These material properties are taken to be homogeneous in each individual layer but may be different from layer to layer. The quantities h_k and h_{k-1} are the distances from the reference surface to the top and bottom surfaces of the kth layer, respectively. Each layer is assumed to have constant thickness.

The appearance of both in-plane strains and changes of curvature in Eqs. (2) and (3) indicates coupling of extensional and flexural effects. This type of coupling has been separated by a perturbation technique.[3, 4] The presence of elastic

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‡ This coupling has been pointed out by Ambartsumyan[2] for laminated anisotropic shells and independently by Reissner and Stavsky[5] for laminated aeolotropic plates.

§ A perturbation method of solution for homogeneous orthotropic plates has been given by Vinson and Brull.[6] By their scheme, successive systems of isotropic plate equations were solved. The solution to the orthotropic plate was then given as the sum of the isotropic plate solutions.

coefficients with subscripts 16 and 26 denotes the most general degree of plane anisotropy. In this analysis, it is assumed that these coefficients are small so that the perturbation technique can be applied.

Perturbation Method Applied to Anisotropic Shells

The assumption that the shells under consideration are only slightly anisotropic suggests that a perturbation method can be employed to obtain a solution. In this method, the general equations are reduced to successive systems of orthotropic shell equations. The solution to the first system is that for an orthotropic shell. Successive corrections are added to account for the general plane anisotropy. To apply this method, the fundamental quantities are parameterized to assess their relative orders of magnitude. Let

$$[A_M,F_M,D_M] = \text{minimums of } [A_{ij},F_{ij},D_{ij}] \quad i,j = 1,2 \text{ and } i = j = 6 \tag{5}$$

$$[A_A,F_A,D_A] = \text{maximums of } [A_{16} \text{ or } A_{26}, F_{16} \text{ or } F_{26}, D_{16} \text{ or } D_{26}] \tag{6}$$

Assume that t is a measure of length such that A_Mt, A_Mt^2, A_At, and A_At^2 are on the order of F_M, D_M, F_A, and D_A, respectively. Furthermore, let

$$A_{ij}^* = A_{ij}/A_M \qquad F_{ij}^* = F_{ij}/A_Mt$$
$$D_{ij}^* = D_{ij}/A_Mt^2 \qquad i,j = 1,2 \text{ and } i = j = 6 \tag{7}$$

$$\begin{aligned} A_{16}^* &= A_{16}/A_A & A_{26}^* &= A_{26}/A_A \\ F_{16}^* &= F_{16}/A_At & F_{26}^* &= F_{26}/A_At \\ D_{16}^* &= D_{16}/A_At^2 & D_{26}^* &= D_{26}/A_At^2 \end{aligned} \tag{8}$$

$$\begin{aligned} N_\alpha^* &= N_\alpha/A_Mt & N_\beta^* &= N_\beta/A_Mt \\ N_{\alpha\beta}^* &= N_{\alpha\beta}/A_Mt & M_\alpha^* &= M_\alpha/A_Mt^2 \\ M_\beta^* &= M_\beta/A_Mt^2 & M_{\alpha\beta}^* &= M_{\alpha\beta}/A_Mt^2 \\ Q_\alpha^* &= Q_\alpha/A_Mt & Q_\beta^* &= Q_\beta/A_Mt \end{aligned} \tag{9}$$

If the degree of anisotropy is small, then A_A is small compared to A_M. Therefore, the quantity δ defined as

$$\delta = A_A/A_M \tag{10}$$

is a small number and will be taken as the perturbation parameter. By introducing Eqs. (5–10) into Eqs. (2) and (3), the following set is obtained:

$$\begin{bmatrix} N_\alpha^* \\ B_\beta^* \\ N_{\alpha\beta}^* \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & \delta A_{16}^* \\ A_{12}^* & A_{22}^* & \delta A_{26}^* \\ \delta A_{16}^* & \delta A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} \epsilon_1^* \\ \epsilon_2^* \\ \gamma_{12}^* \end{bmatrix} + \begin{bmatrix} F_{11}^* & F_{12}^* & \delta F_{16}^* \\ F_{12}^* & F_{22}^* & \delta F_{26}^* \\ \delta F_{16}^* & \delta F_{26}^* & F_{66}^* \end{bmatrix} \begin{bmatrix} \chi_1^* \\ \chi_2^* \\ 2\chi_{12}^* \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} M_\alpha^* \\ M_\beta^* \\ M_{\alpha\beta}^* \end{bmatrix} = \begin{bmatrix} F_{11}^* & F_{12}^* & \delta F_{16}^* \\ F_{12}^* & F_{22}^* & \delta F_{26}^* \\ \delta F_{16}^* & \delta F_{26}^* & F_{66}^* \end{bmatrix} \begin{bmatrix} \epsilon_1^* \\ \epsilon_2^* \\ \gamma_{12}^* \end{bmatrix} + \begin{bmatrix} D_{11}^* & D_{12}^* & \delta D_{16}^* \\ D_{12}^* & D_{22}^* & \delta D_{26}^* \\ \delta D_{16}^* & \delta D_{26}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} \chi_1^* \\ \chi_2^* \\ 2\chi_{12}^* \end{bmatrix} \tag{12}$$

where ϵ_1^*, ϵ_2^*, γ_{12}^* and χ_1^*, χ_2^*, χ_{12}^* are the parameterized in-plane strain components and the changes of curvature, respectively. In terms of the parameterized reference surface displacements u_0^*, v_0^*, and w^*, ϵ_1^*, . . . and χ_1^*, . . . possess the identical forms as that of homogeneous shell theory (see, for example, Ref. 7, p. 24). In these strain-displacement and curvature-displacement relations, the displacements, the surface coordinates, and the radii of curvature are parameterized in the following way:

$$\begin{aligned} u_0^* &= u_0/t^2 \qquad v_0^* = v_0/t^2 \qquad w^* = w/t^2 \\ \alpha^* &= \alpha/t \qquad \beta^* = \beta/t \\ R_1^* &= R_1/t \qquad R_2^* = R_2/t \end{aligned} \tag{13}$$

Equilibrium equations and the equation of compatibility on the reference surface likewise do not change in form from that of homogeneous shell theory; however, the quantities appearing in them are parameterized. The loads appearing in the equilibrium equations are parameterized according to

$$q_\alpha^* = q_\alpha/A_M \qquad q_\beta^* = q_\beta/A_M \qquad q_z^* = q_z/A_M \tag{14}$$

The boundary conditions associated with the governing system of equations are identical to those of isotropic homogeneous shell theory. However, they appear in parameterized form and must be applied at the selected reference surface instead of at the middle surface.

Equations for Cylindrical Shells

In this section the general equations that were previously derived are specialized for circular cylindrical shells. The well-known Donnell approximations are employed to arrive at the governing equations. For a cylindrical shell, the following coordinate system has been adopted:

$$\begin{aligned} X &= a^*t\cos(y/a^*) \\ Y &= a^*t\sin(y/a^*) \\ Z &= xt \end{aligned} \tag{15}$$

where

$$a^* = a/t \qquad x = x'/t \qquad y = y'/t \tag{16}$$

with x' and y' being the unparameterized (natural) coordinates of the reference surface. For this system of coordinates, the surface metric coefficients are $A = 1$ and $B = 1$, and the radii of curvature are $R_1 = \infty$ and $R_2 = a$.

The strain-displacement and curvature change-displacement relations, in parameterized form incorporating Donnell's approximations, are[‖]

$$\epsilon_1^* = u_{0,x}^* \qquad \epsilon_2^* = v_{0,y}^* \qquad \gamma_{12}^* = u_{0,y}^* + v_{0,x}^* \tag{17}$$

$$\chi_1^* = -w_{,xx}^* \qquad \chi_2^* = -w_{,yy}^* \qquad \chi_{12}^* = -w_{,xy}^* \tag{18}$$

The equilibrium and compatibility equations for cylindrical shells take the forms

$$\left.\begin{aligned} N_{x,x}^* + N_{xy,y}^* &= 0 \\ N_{xy,x}^* + N_{y,y}^* &= 0 \\ Q_{x,x}^* + Q_{y,y}^* - N_y^*/a^* + q_z^* &= 0 \\ M_{x,x}^* + M_{xy,y}^* - Q_x^* &= 0 \\ M_{xy,y}^* + M_{y,y}^* - Q_y^* &= 0 \end{aligned}\right\} \tag{19}$$

$$\epsilon_{2,xx}^* + \epsilon_{1,yy}^* - \gamma_{12,xy}^* - w_{,xx}^*/a^* = 0 \tag{20}$$

In the first two equations of (19), q_x^* and q_y^* are taken to be zero, and Donnell's approximations are invoked.

For this problem it is expedient to regard the transverse deflection w^* and the in-plane forces N_x^*, N_y^*, N_{xy}^* as the primary dependent variables. The introduction of the Airy stress function U^* in parameterized form further reduces the number of variables to two:

$$N_x^* = U_{,yy}^* \qquad N_y^* = U_{,xx}^* \qquad N_{xy}^* = -U_{,xy}^* \tag{21}$$

where

$$U^* = U/A_Mt^3 \tag{22}$$

Equation (21) satisfies the first two equations of (19) identically. Substituting the fourth and fifth equations into the third equation of (19) yields

$$M_{x,xx}^* + 2M_{xy,xy}^* + M_{y,yy}^* - N_y^*/a^* + q_z^* = 0 \tag{23}$$

‖ A comma in the subscripts denotes partial differentiation with respect to the variables that follow.

Equations (20) and (23) constitute the governing system of equations for the problem. They can be expressed in terms of U^* and w^* by inverting Eq. (11):

$$[\epsilon^*] = [B^*][N^*] - [B^*][F^*][\chi^*] \tag{24}$$

where the matrix $[B^*]$ is the inverse of $[A^*]$. Substitution of Eq. (24) into Eq. (12) gives

$$[M^*] = [b^*][N^*] + [d^*][\chi^*] \tag{25}$$

where

$$\begin{aligned} [b^*] &= [F^*][B^*] \\ [d^*] &= [D^*] - [F^*][B^*][F^*] \end{aligned} \tag{26}$$

As the perturbation parameter δ appears within the matrices $[A^*]$, $[F^*]$, and $[D^*]$, the matrix operations yielding $[B^*]$, $[b^*]$, and $[d^*]$ have the form

$$[\lambda] = \begin{bmatrix} \sum_{n=0}^{\infty} \lambda_{11}^{(n)}\delta^{2n} & \sum_{n=0}^{\infty} \lambda_{12}^{(n)}\delta^{2n} & \sum_{n=0}^{\infty} \lambda_{16}^{(n)}\delta^{2n+1} \\ \sum_{n=0}^{\infty} \lambda_{21}^{(n)}\delta^{2n} & \sum_{n=0}^{\infty} \lambda_{22}^{(n)}\delta^{2n} & \sum_{n=0}^{\infty} \lambda_{26}^{(n)}\delta^{2n+1} \\ \sum_{n=0}^{\infty} \lambda_{61}^{(n)}\delta^{2n+1} & \sum_{n=0}^{\infty} \lambda_{62}^{(n)}\delta^{2n+1} & \sum_{n=0}^{\infty} \lambda_{66}^{(n)}\delta^{2n} \end{bmatrix} \lambda = B^*, b^*, d^* \tag{27}$$

But, since δ is considered a small quantity, only the leading coefficient in λ will be retained. Thus

$$[\lambda] = \begin{bmatrix} \lambda_{11}^{(0)} & \lambda_{12}^{(0)} & \delta\lambda_{16}^{(0)} \\ \lambda_{21}^{(0)} & \lambda_{22}^{(0)} & \delta\lambda_{26}^{(0)} \\ \delta\lambda_{61}^{(0)} & \delta\lambda_{62}^{(0)} & \lambda_{66}^{(0)} \end{bmatrix} \lambda = B^*, b^*, d^* \tag{28}$$

Equations (24) and (25), written explicitly, are

$$\left.\begin{aligned} \epsilon_1^* &= B_{11}^*U_{,yy}^* + B_{12}^*U_{,xx}^* + b_{11}^*w_{,xx}^* + \\ &\quad b_{21}^*w_{,yy}^* + \delta[-B_{16}^*U_{,xy}^* + 2b_{61}^*w_{,xy}^*] \\ \epsilon_2^* &= B_{12}^*U_{,yy}^* + B_{22}^*U_{,xx}^* + b_{12}^*w_{,xx}^* + \\ &\quad b_{22}^*w_{,yy}^* + \delta[-B_{26}^*U_{,xy}^* + 2b_{62}^*w_{,xy}^*] \\ \gamma_{12}^* &= -B_{66}^*U_{,xy}^* + 2b_{66}^*w_{,xy}^* + \\ &\quad \delta[B_{16}^*U_{,yy}^* + B_{26}^*U_{,xx}^* + b_{16}^*w_{,xx}^* + b_{26}^*w_{,yy}^*] \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} M_x^* &= b_{11}^*U_{,yy}^* + b_{12}^*U_{,xx}^* - d_{11}^*w_{,xx}^* - \\ &\quad d_{12}^*w_{,yy}^* + \delta[-b_{16}^*U_{,xy}^* - 2d_{16}^*w_{,xy}^*] \\ M_y^* &= b_{21}^*U_{,yy}^* + b_{22}^*U_{,xx}^* - d_{12}^*w_{,xx}^* - \\ &\quad d_{22}^*w_{,yy}^* + \delta[-b_{26}^*U_{,xy}^* - 2d_{26}^*w_{,xy}^*] \\ M_{xy}^* &= -b_{66}^*U_{,xy}^* - 2d_{66}^*w_{,xy}^* + \\ &\quad \delta[b_{61}^*U_{,yy}^* + b_{62}^*U_{,xx}^* - d_{16}^*w_{,xx}^* - d_{26}^*w_{,yy}^*] \end{aligned}\right\} \tag{30}$$

These expressions appear as orthotropic shell equations with small corrective terms to account for the general anisotropy. Substitution of Eqs. (29) and (30) into Eqs. (20) and (23) gives

$$L_1U^* + L_2w^* - w_{,xx}^*/a^* = \delta[L_4U^* - L_5w^*] \tag{31}$$

$$L_2U^* - L_3w^* - U_{,xx}^*/a^* + q_z^* = \delta[L_6w^* - L_5U^*] \tag{32}$$

where

$$\begin{aligned} L_1\{\ \} &= B_{22}^*\{\ \}_{,xxxx} + (2B_{12}^* + B_{66}^*)\{\ \}_{,xxyy} + \\ &\quad B_{11}^*\{\ \}_{,yyyy} \\ L_2\{\ \} &= b_{12}^*\{\ \}_{,xxxx} + (b_{11}^* + b_{22}^* - 2b_{66}^*) \times \\ &\quad \{\ \}_{,xxyy} + b_{21}^*\{\ \}_{,yyyy} \\ L_3\{\ \} &= d_{11}^*\{\ \}_{,xxxx} + 2(d_{12}^* + 2d_{66}^*)\{\ \}_{,xxyy} + \\ &\quad d_{22}^*\{\ \}_{,yyyy} \\ L_4\{\ \} &= 2B_{26}^*\{\ \}_{,xxxy} + 2B_{16}^*\{\ \}_{,xyyy} \\ L_5\{\ \} &= (2b_{62}^* - b_{16}^*)\{\ \}_{,xxxy} + \\ &\quad (2b_{61}^* - b_{26}^*)\{\ \}_{,xyyy} \\ L_6\{\ \} &= 4d_{16}^*\{\ \}_{,xxxy} + 4d_{26}^*\{\ \}_{,xyyy} \end{aligned} \tag{33}$$

The left-hand sides of Eqs. (31) and (32) are the orthotropic shell equations, and the right-hand sides involve the corrective terms for general anisotropy.

Let the solution to Eqs. (31) and (32) be taken as a power series expansion in δ for both U^* and w^*:

$$\begin{aligned} w^* &= w_1^* + \delta w_2^* + \delta^2 w_3^* + \ldots \\ U^* &= U_1^* + \delta U_2^* + \delta^2 U_3^* + \ldots \end{aligned} \tag{34}$$

Then Eqs. (31) and (32) reduce to the following systems of equations which will be solved successively:

First System

$$\begin{aligned} L_1U_1^* + L_2w_1^* - w_{1,xx}^*/a^* &= 0 \\ L_2U_1^* - L_3w_1^* - U_{1,xx}^*/a^* + q_z^* &= 0 \end{aligned} \tag{35}$$

δ^n System ($n = 1,2,\ldots$)

$$\begin{aligned} L_1U_{n+1}^* + L_2w_{n+1}^* - w_{n+1,xx}^*/a^* &= L_4U_n^* - L_5w_n^* \\ L_2U_{n+1}^* - L_3w_{n+1}^* - U_{n+1,xx}^*/a^* &= L_6w_n^* - L_5U_n^* \end{aligned} \tag{36}$$

These systems of equations must be solved with the appropriate boundary conditions. For a complete cylindrical shell, these conditions to be evaluated at x = const are

$$\begin{aligned} M_{xy}^* + M_{xy}^*/a^* &= h_1^*(y) \\ N_x^* &= h_2^*(y) \\ Q_x^* + M_{xy,y}^* &= h_3^*(y) \\ M_x^* &= h_4^*(y) \end{aligned} \tag{37}$$

where h_1^* through h_4^* are prescribed functions of y in parameterized form defined by

$$\begin{aligned} h_i^*(y) &= h_i(y)/A_M t \qquad i = 1,2,3 \\ h_4^*(y) &= h_4(y)/A_M t^2 \end{aligned} \tag{38}$$

In terms of U^* and w^*, Eq. (37) becomes

$$\begin{aligned} &-(1 + b_{66}^*/a^*)U_{,xy}^* - (2d_{66}^*/a^*)w_{,xy}^* + \\ &\quad (\delta/a^*)[b_{61}^*U_{,yy}^* + b_{62}^*U_{,xx}^* - d_{16}^*w_{,xx}^* - \\ &\quad d_{26}^*w_{,yy}^*] = h_1^*(y) \\ &U_{,yy}^* = h_2^*(y) \\ &b_{12}^*U_{,xxx}^* + (b_{11}^* - 2b_{66}^*)U_{,xyy}^* - d_{11}^*w_{,xxx}^* - \\ &\quad (d_{12}^* + 4d_{66}^*)w_{,xyy}^* + \delta[2b_{61}^*U_{,yyy}^* + (2b_{62}^* - b_{16}^*) \times \\ &\quad U_{,xxy}^* - 4d_{16}^*w_{,xxy}^* - 2d_{26}^*w_{,yyy}^*] = h_3^*(y) \\ &b_{11}^*U_{,yy} + b_{12}^*U_{,xx}^* - d_{11}^*w_{,xx}^* - d_{12}^*w_{,yy}^* - \\ &\quad \delta[b_{16}^*U_{,xy}^* + 2d_{16}^*w_{,xy}^*] = h_4^*(y) \end{aligned} \tag{39}$$

For the solution form given by Eq. (34), the following systems of boundary conditions must be prescribed to the corresponding systems of differential equations:

First System

$$\begin{aligned} &-(1 + b_{66}^*/a^*)U_{1,xy}^* - (2d_{66}^*/a^*)w_{1,xy}^* = h_1^*(y) \\ &U_{1,yy}^* = h_2^*(y) \\ &(b_{11}^* - 2b_{66}^*)U_{1,xyy}^* + b_{12}^*U_{1,xxx}^* - d_{11}^*w_{1,xxx}^* - \\ &\quad (d_{12}^* + 4d_{66}^*)w_{1,xyy}^* = h_3^*(y) \\ &b_{11}^*U_{1,yy}^* + b_{12}^*U_{1,xx}^* - d_{11}^*w_{1,xx}^* - d_{12}^*w_{1,yy}^* = h_4^*(y) \end{aligned} \tag{40}$$

δ^n System ($n = 1,2\ldots$)

$$(a^* + b_{66}^*)U_{n+1,xy}^* - 2d_{66}^*w_{n+1,xy}^* = b_{61}^*U_{n,yy}^* + b_{62}^*U_{n,xx}^* - d_{16}^*w_{n,xx}^* - d_{26}^*w_{n,yy}^*$$

$$U_{n+1,yy} = 0$$

$$(b_{11}^* - 2b_{66}^*)U_{n+1,xyy}^* + b_{12}^*U_{n+1,xxx}^* - d_{11}^*w_{n+1,xxx}^* - (d_{12}^* + 4d_{66}^*)w_{n+1,xyy}^* = 4d_{16}w^*_{n,xxy} + 2d_{26}w^*_{n,yyy} - 2b_{61}U^*_{n,yyy} - (2b_{62} - b_{16})U_{n,xxy} \quad (41)$$

$$b_{11}^*U_{n+1,yy}^* + b_{12}^*U_{n+1,xx}^* - d_{11}^*w_{n+1,xx}^* - d_{12}^*w_{n+1,yy}^* = b_{16}^*U_{n,xy}^* + 2d_{16}^*w_{n,xy}^*$$

To illustrate the solution method for cylindrical shells, the following problem is presented. Consider a semi-infinite cylindrical shell♯ that is subjected to a uniform pressure of intensity q_0^* and to a constant extensional force N_{xc}^*. Since the loading conditions do not depend on the y coordinate, the first system of equations will degenerate into a coupled system of ordinary differential equations. The boundary conditions at $x = 0$ for this case are taken as

$$\begin{aligned} N_x^* &= N_{xc}^* \\ N_{xy}^* + M_{xy}^*/a^* &= 0 \\ Q_x^* &= Q_0^* \\ M_x^* &= M_0^* \end{aligned} \quad (42)$$

The variable U_1^* of the first system may be eliminated by integrating the first equation of (35) twice and substituting the result into the second equation, which for discussion purposes will be called the determinative equation. The constants associated with the integral of the first equation, which are akin to N_x^*, have been evaluated to satisfy axisymmetrical geometry. The solution of the determinative equation is

$$w_1^*(x) = \exp(-\lambda\gamma x)[k_{11}\cos\mu\gamma x + k_{12}\sin\mu\gamma x] \times \exp(\lambda\gamma x)[k_{13}\cos\mu\gamma x + k_{14}\sin\mu\gamma x] + (a^*)^2B_{22}^*q_0^* + B_{12}^*a^*N_{xc}^* \quad (43)$$

where λ, μ, and γ are the roots of the auxiliary algebraic equation associated with the homogeneous determinative equation and are related to the elastic coefficients in the following manner:

$$\gamma = [(a^*)^2\{d_{11}^*B_{22}^* + (b_{12}^*)^2\}]^{-1/4}$$
$$\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix} = \begin{Bmatrix}\text{Real}\\ \text{Im}\end{Bmatrix}\left[\gamma\{a^*(b_{12}^* + i[d_{11}^*B_{22}^*]^{1/2})\}^{1/2}\right] \quad (44)$$

where $i = -1^{1/2}$. Evaluating k_{11} through k_{14} from conditions (42) gives

$$k_{11} = (1/\Delta)[2Q_0^*\lambda\mu\gamma^2C_1 - (M_0^* - b_{11}^*N_{xc}^*) \times \{C_2\mu\gamma - \gamma^3\mu(3\lambda^2 - \mu^2)C_1\}]$$

$$k_{12} = (1/\Delta)[(M_0^* - b_{11}^*N_{xc}^*) \times \{C_2\lambda\gamma + \gamma^3\lambda(3\mu^2 - \lambda^2)C_1\} + Q_0^*\{C_2 - \gamma^2(\lambda^2 - \mu^2)C_1\}] \quad (45)$$

$$k_{13} = k_{14} = 0$$

where

$$\Delta = [-C_2\lambda\gamma - \gamma^3\lambda(3\mu^2 - \lambda^2)C_1][2\lambda\mu\gamma^2C_1] - [C_2 - \gamma^2(\lambda^2 - \mu^2)C_1][C_2\mu\gamma - \lambda^3\mu(3\lambda^2 - \mu^2)C_1] \quad (46)$$

and

$$C_1 = d_{11}^* + (b_{12}^*)^2/B_{22}^*$$

$$C_2 = b_{12}^*/B_{22}^*a^*$$

Knowing $w_1^*(x)$, it requires only a straightforward calculation to obtain U_1^*.

With the solution of the first system, the right-hand sides of the second system of governing equations and its boundary conditions may be calculated. The nature of the solution to the first system results in the vanishing of the right-hand sides of the differential equations and some of the boundary conditions for the second system. The only remaining nonhomogeneous boundary condition for the second system is

$$(a^* + b_{66}^*)U_{2,xy}^* + 2d_{66}^*w_{2,xy}^* = b_{61}^*N_{xc}^* + \frac{b_{62}^*}{B_{22}^*}\left[\frac{k_{11} + (a^*)^2B_{22}^*q_0^*}{a^*} - k_{11}\gamma^2(\lambda^2 - \mu^2) + 2k_{12}\lambda\mu\gamma^2\right] + d_{16}^*[k_{11}^*\gamma^2(\lambda^2 - \mu^2) - 2k_{12}\lambda\mu\gamma^2] \quad (47)$$

The functions U_2^* and w_2^* are again required to remain finite as x tends to infinity. Note that the anisotropy of the material is reflected only in a shearing force at the edge $x = 0$ for the second system. Let the solution to the second system be taken as

$$\begin{aligned} U_2^*(x,y) &= \sum_{n=1}^{\infty} {}_2f_n(x)\cos\left(\frac{ny}{a^*}\right) \\ w_2^*(x,y) &= \sum_{n=1}^{\infty} {}_2g_n(x)\cos\left(\frac{ny}{a^*}\right) \end{aligned} \quad (48)$$

Substitution of Eq. (48) into Eq. (36) yields a coupled system of ordinary differential equations with constant coefficients for the functions ${}_2g_n(x)$ and ${}_2f_n(x)$. This resulting system may be uncoupled by raising it to one equation of the eighth order. The solution to the resulting eighth-order equation is

$$\begin{aligned} {}_2f_n(x) &= {}_2A_{n(1)}\exp(xp_n) + {}_2A_{n(2)}\exp(x\bar{p}_n) + {}_2A_{n(3)}\exp(xq_n) + {}_2A_{n(4)}\exp(x\bar{q}_n) \\ {}_2g_n(x) &= {}_2B_{n(1)}\exp(xp_n) + {}_2B_{n(2)}\exp(k\bar{p}_n) + {}_2B_{n(3)}\exp(xq_n) + {}_2B_{n(4)}\exp(x\bar{q}_n) \end{aligned} \quad (49)$$

where p_n, $\bar{p}_n$, q_n, $\bar{q}_n$ are the roots and complex conjugate roots with negative real parts to the auxiliary equation.** The roots with the positive real parts have been omitted because of boundedness conditions at $x = \infty$. The constants of integration ${}_2A_{n(i)}$ and ${}_2B_{n(i)}$ are related in the following manner:

$${}_2A_{n(1)} = \left[\frac{B_{22}^*p_n^4 + C_3p_n^2 + B_{11}^*(n/a^*)^4}{b_{12}^*p_n^4 + C_4p_n^2 + b_{21}^*}\right]{}_2B_{n(1)} = M_1({}_2B_{n(1)})$$

$${}_2A_{n(3)} = \left[\frac{B_{22}^*q_n^4 + C_3q_n^2 + B_{11}^*(n/a^*)^4}{b_{12}^*q_n^4 + C_4q_n^2 + b_{21}^*}\right]{}_2B_{n(3)} = M_2({}_2B_{n(3)}) \quad (50)$$

$${}_2A_{n(2)} = \overline{{}_2A_{n(1)}} \qquad {}_2A_{n(4)} = \overline{{}_2A_{n(3)}}$$

where

$$\begin{aligned} C_3 &= -1/a^* - (b_{11}^* + b_{22}^* - 2b_{66}^*)(n/a^*)^2 \\ C_4 &= (2B_{12}^* + B_{66}^*)(n/a^*)^2 \end{aligned} \quad (51)$$

Substitution of the solution into the boundary conditions leads to the following matrix equation for the evaluation of the constants:

$$\begin{bmatrix} S_A & \bar{S}_A & S_B & \bar{S}_B \\ S_C & \bar{S}_C & S_D & \bar{S}_D \\ S_E & \bar{S}_E & S_F & \bar{S}_F \\ S_G & \bar{S}_G & S_H & \bar{S}_H \end{bmatrix}\begin{bmatrix} {}_2B_{n(1)} \\ {}_2B_{n(2)} \\ {}_2B_{n(3)} \\ {}_2B_{n(4)} \end{bmatrix} = \begin{bmatrix} D_n \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (52)$$

♯ A more detailed discussion of this problem is contained in Ref. 8 and is available on request.

** A bar over a quantity denotes its complex conjugate.

where

$$S_A = (n/2)[M_1(1 + b_{66}^*/a^*) + 2d_{66}^*/a^*]p_n$$
$$S_B = (n/2)[M_2(1 + b_{66}^*/a^*) + 2d_{66}/a^*]q_n$$
$$S_C = -M_1 n^2/4 \qquad S_D = -M_2 n^2/4$$
$$S_E = p^3{}_n(b_{12}^*M_1 - d_{11}^*) + (p_n/4)[d_{12}^* + 4d_{66}^* - M_1(b_{11}^* - 2b_{66}^*)]n^2 \quad (53)$$
$$S_F = q_n^3(b_{12}^*M_2 - d_{11}^*) + (q_n/4)[d_{12}^* + 4d_{66}^* - M_2(b_{11}^* - 2b_{66}^*)]n^2$$
$$S_G = p_n^2(b_{12}^*M_1 - d_{11}^*) + (n^2/4)(d_{12}^* - b_{11}^*M_1)$$
$$S_H = q_n^2(b_{12}^*M_2 - d_{11}^*) + (n^2/4)(d_{12}^* - b_{11}^*M_2)$$

The D_n's are the Fourier coefficients of the expansion of the right-hand side of Eq. (47) which will be denoted by K. Hence,

$$D_n = \int_0^\pi K \sin\left(\frac{ny}{a^*}\right) dy \Big/ \left[\int_0^\pi \sin^2\left(\frac{ny}{a^*}\right) dy\right] \quad (54)$$

In the same manner, the third and subsequent systems of equations may be solved. The boundary conditions depend on the solution of the previous system.

After a sufficient number of systems have been solved, the solution to the complete problem must be reconstituted by summing the solutions of the individual systems and returning to the original parameters by means of Eqs. (8, 9, 13, and 14).

Discussion

A perturbation method of solution has been applied to laminated anisotropic shells. By this scheme the effect of general anisotropy was reduced to orthotropy such that the solution to an anisotropic shell problem consists of a series of orthotropic shell solutions. The method was demonstrated for the uniform pressurization of laminated cylindrical shells. Although the method of solution is straightforward, the amount of algebra involved is quite extensive. Therefore, it is suggested that this method be used on shell structures where the general anisotropy is slight. Then a good approximate solution may be obtained by solving a small number of systems of equations.

By letting the radius of curvature a tend to infinity throughout the equations for cylindrical shells, a system of laminated anisotropic plate equations is obtained. Techniques employed previously for cylindrical shells are again applicable for laminated anisotropic plates.

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A Transient Axisymmetric Thermoelastic Problem for the Hollow Sphere

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The linear uncoupled quasi-static theory of thermoelasticity has been applied to the elastic hollow sphere having a prescribed axisymmetric transient heat input on the outside surface, and a prescribed axisymmetric transient temperature distribution on the inside surface. Series expressions for the temperature, stress, and displacement fields are obtained in terms of orthogonal functions. As a particular example of the analysis, the stresses due to aerodynamic heating of a hypersonic hollow sphere are investigated in detail, and some representative transient and steady-state stress distributions are presented in graphical and tabular form.

Introduction

The thermoelastic problem for a sphere has been the object of numerous investigations in the past, and bibliographies of this work may be found in the references.[1–3] Of particular interest to the present investigation is the work of Trostel,[4] who obtains quite general solutions for general transient axisymmetric boundary conditions on the hollow sphere. The corresponding steady-state problem has been solved by McDowell and Sternberg,[5] and a solution for the transient thermal stresses in a solid sphere has been obtained by Melan.[6] The purpose of the present investigation is to present a detailed analysis of the transient thermal stresses in a hollow sphere subject to a prescribed axisymmetric transient heat input on the outside surface, and to a prescribed axisymmetric transient temperature distribution on the inside surface. The inner and outer surfaces are assumed

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